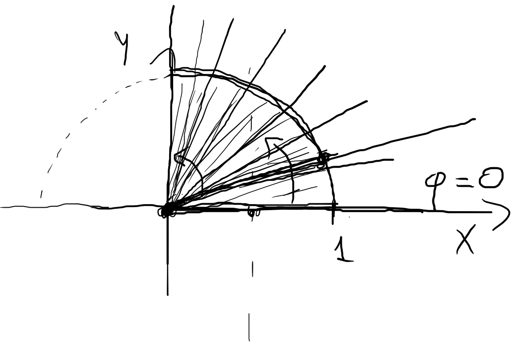
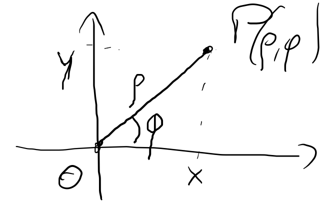
1 Použitím polárních souřadnic spočítejte integrály $\underbrace{\int_0^1 \int_0^{\sqrt{1-x^2}}}_{A} \underbrace{\arctan \frac{y}{x}}_{\text{omezená}}\, dy\, dx$.





$$\arctan \frac{y}{x} = \operatorname{arctg} \frac{y}{x}$$

$$\operatorname{arctg} \frac{\rho \sin\varphi}{\rho\cos\varphi} = \operatorname{arctg}(\operatorname{tg}\varphi) = \varphi$$

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$

$$\vec{\Phi}(\rho,\varphi) = (\underbrace{\rho\cos\varphi}_{\Phi_1}, \underbrace{\rho\sin\varphi}_{\Phi_2})$$

$$\vec{\Phi}(U) = A$$

$$J_{\vec{\Phi}} = \left|\det\begin{pmatrix} \Phi_{1\rho} & \Phi_{1\varphi} \\ \Phi_{2\rho} & \Phi_{2\varphi}\end{pmatrix}\right| = \underline{\underline{\rho}} \quad \text{Jakobián}$$

$$y = \sqrt{1-x^2} \iff \begin{cases} y \geq 0 \\ -1 \leq x \leq 1 \\ \underline{\underline{x^2+y^2 = 1}} \end{cases} \rightsquigarrow \rho^2 = 1 \Rightarrow \underline{\underline{\rho = 1}}$$

$$\int_0^{\pi/2}\int_0^1 \varphi \cdot \rho \, d\rho\, d\varphi =$$
$$= \int_0^{\pi/2} \varphi\, d\varphi \cdot \int_0^1 \rho\, d\varphi = \left[\frac{\varphi^2}{2}\right]_0^{\pi/2} \cdot \left[\frac{\rho^2}{2}\right]_0^1 = \frac{1}{2}\,\frac{\pi^2}{8} = \frac{\pi^2}{16}$$

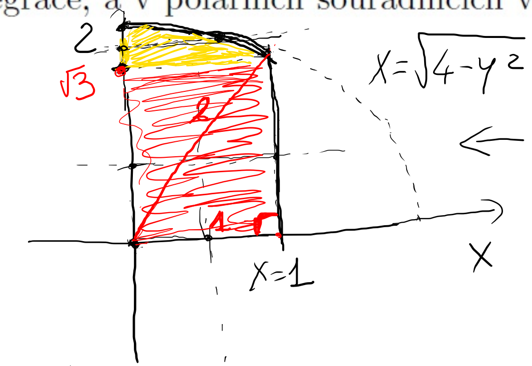
$$\boxed{\int_a^b\int_c^d f(x)\cdot g(y)\, dx\, dy = \int_a^b g(y)\, dy \cdot \int_c^d f(x)\, dx}$$

2 Přepište následující integrál $\int_0^1 \int_0^{\sqrt{4-x^2}} f\,dy\,dx$ v opačném pořadí integrace, a v polárních souřadnicích v pořadí $d\varrho\, d\varphi$.

$$\int_0^1 \int_0^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

$$\int_0^{\sqrt{3}} \int_0^1 f(x,y)\,dx\,dy + \int_{\sqrt{3}}^2 \int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$$

$$\int_0^{\pi/3} \int_0^{\frac{1}{\cos\varphi}} f(\rho\cos\varphi, \rho\sin\varphi)\cdot\rho\, d\rho\, d\varphi + \int_{\pi/3}^{\pi/2} \int_0^2 f(\rho\cos\varphi, \rho\sin\varphi)\rho\, d\rho\, d\varphi$$

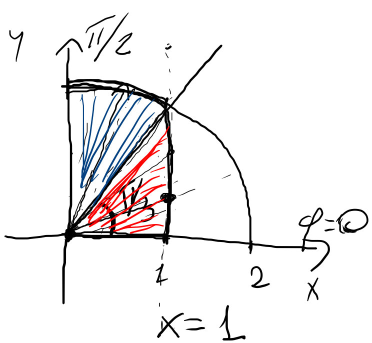


$y = \sqrt{4-x^2}$

$$\begin{cases} y \geq 0 \\ -2 \leq x \leq 2 \\ x^2+y^2=4 \end{cases}$$

$\hookrightarrow x^2 = 4-y^2$

$x = \pm\sqrt{4-y^2}$



$x=1 \underset{pol.}{\rightsquigarrow} \rho\cos\varphi = 1 \quad \rho = \frac{1}{\cos\varphi}$

$x^2+y^2=4 \underset{pol.}{\rightsquigarrow} \rho = 2$

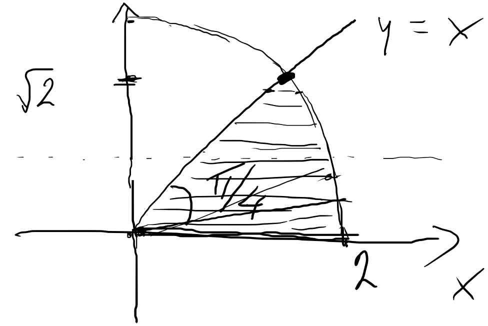
$$\int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy.$$

$$\int_0^{\pi/4} \int_0^2 \frac{1}{1+\rho^2}\, \rho \, d\rho \, d\varphi =$$

$$= \int_0^{\pi/4} 1 \, d\varphi \cdot \int_0^2 \frac{1}{2} \frac{2\rho}{1+\rho^2}\, d\rho$$

$u = 1+\rho^2$
$du = 2\rho\, d\rho$

$$= \pi/4 \cdot \left[\frac{\ln(1+\rho^2)}{2} \right]_0^2 = \frac{\pi}{8} \ln 5$$



$x = y$
$x = \sqrt{4-y^2}$

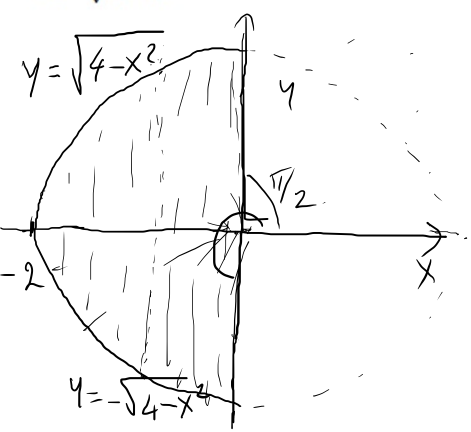
$$\begin{cases} x \geq 0 \\ -2 \leq y \leq 2 \end{cases}$$

$x^2 + y^2 = 4$
$x^2 + (\sqrt{2})^2 = 4$
$x^2 = 2$

$\rho = 2$

$\sqrt{\rho^2} = \rho \quad \rho \geq 0$

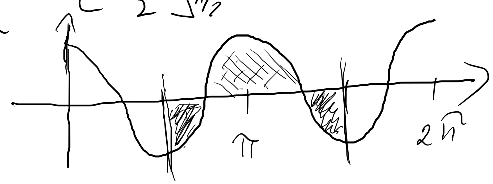
$$\int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{x^2-y^2}{\sqrt{x^2+y^2}}\, dy\, dx.$$



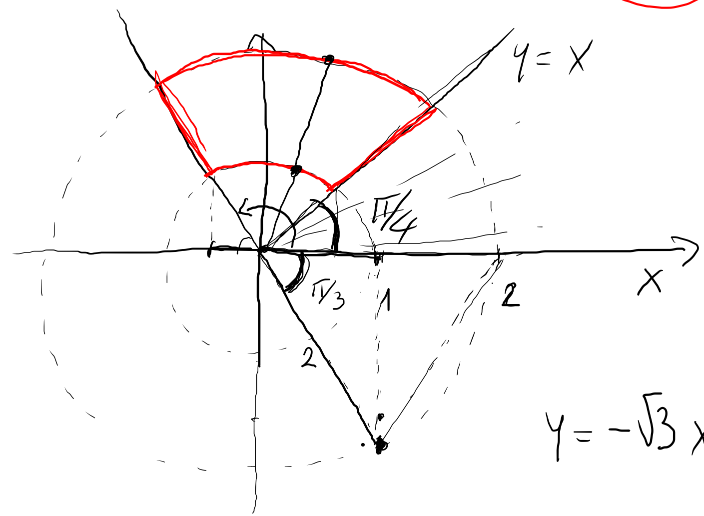
$$\int_{\pi/2}^{3/2\pi} \int_0^2 \frac{\rho^2\cos^2\varphi - \rho^2\sin^2\varphi}{\cancel{\sqrt{\rho^2}}}\, \cancel{\rho}\, d\rho\, d\varphi =$$

$$= \int_{\pi/2}^{3/2\pi} \cos 2\varphi \, d\varphi \cdot \int_0^2 \rho^2 \, d\rho = 0$$

$$\left[\frac{\sin 2\varphi}{2}\right]_{\pi/2}^{3/2\pi}$$



Spočítejte $\iint\limits_D \frac{y}{\sqrt{x^2+y^2}}\,dA$, kde D je omezeno křivkami $x^2+y^2=1$, $x^2+y^2=4$, $y=-\sqrt{3}x$, $y=x$ a $y \geq 0$.



$$\int_{\pi/4}^{\frac{2}{3}\pi}\int_1^2 \frac{\rho \sin\varphi}{\cancel{\rho}}\,\cancel{\rho}\,d\rho\,d\varphi =$$

$$= \int_{\pi/4}^{\frac{2}{3}\pi}\left(\frac{\rho^2}{2}\sin\varphi\right)_{\rho=1}^{\rho=2} d\varphi =$$

$$= \int_{\pi/4}^{\frac{2}{3}\pi}\left(2-\frac{1}{2}\right)\cdot\sin\varphi\,d\varphi = \frac{3}{2}\left[-\cos\varphi\right]_{\pi/4}^{\frac{2}{3}\pi} = \frac{3}{2}\left(\frac{1}{2}+\frac{\sqrt{2}}{2}\right)$$

S použitím substituce spočítejte $\int\limits_0^1 \int\limits_0^{1-x} \sqrt{x+y}(y-2x)^2 \; dA.$ $dy\,dx$

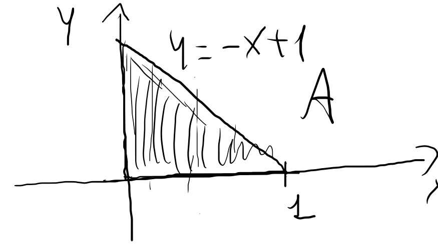
$$\left(\begin{array}{cc|c} 1 & 1 & u \\ -2 & 1 & v \end{array}\right) \sim \dots$$

ϕ^{-1} $\quad u = x+y$

$\qquad v = y-2x$

$$\phi^{-1}(x,y) = (x+y,\; y-2x)$$

$$\phi^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$$



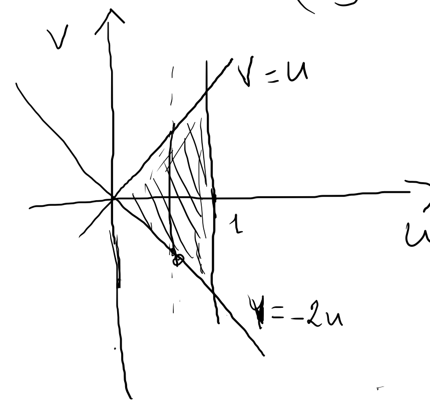
$$\int_0^1 \int_{-2u}^{u} \sqrt{u} \cdot v^2 \; \Big|\underbrace{\det J_{\vec{\phi}}}_{1/3}\Big| \, dv \, du =$$

$$= \frac{1}{3} \int_0^1 \sqrt{u} \left[\frac{v^3}{3}\right]_{-2u}^{u} du = \int \frac{1}{\cancel{3}} \cancel{3} \underbrace{u^3 \sqrt{u}}_{u^{7/2}} \, du =$$

$$= \left[u^{9/2} \cdot \frac{2}{9}\right]_0^1 = \frac{2}{9}$$

$\phi^{-1} \downarrow \qquad \uparrow \phi(u,v)$

$$\phi(u,v) = \left(\frac{u-v}{3}, \frac{2u+v}{3}\right)$$



$y=0 \rightsquigarrow v=-2u$

$x=0 \rightsquigarrow v=u$

$x+y=1 \rightsquigarrow u=1$

$$\left|\det J_{\phi^{-1}}\right| = \left|\det \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}\right| = 3$$

$$\left|\det J_{\vec{\phi}}\right| = \frac{1}{3}$$

$$\left|\det J_{\vec{\phi}}\right| = \frac{1}{\left|\det J_{\phi^{-1}}\right|}$$

$$\begin{cases} u = x+y \\ v = y-2x \end{cases}$$

$$\left(\begin{array}{cc|c} 1 & 1 & u \\ -2 & 1 & v \end{array}\right) \overset{2R_1+R_2}{\sim} \left(\begin{array}{cc|c} 1 & 1 & u \\ 0 & 3 & 2u+v \end{array}\right)$$

$$x = u - \frac{2u+v}{3} = \frac{u-v}{3}$$

$$y = \frac{2u+v}{3}$$

$$\vec{\phi}(u,v) = \left(\frac{u-v}{3}, \frac{2u+v}{3} \right)$$

$$|\det J\phi| = \left| \det \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix} \right| = \frac{2}{9} + \frac{2}{3} = \frac{3}{9} = \frac{1}{3} = \frac{1}{|\det J\phi^{-1}|}$$

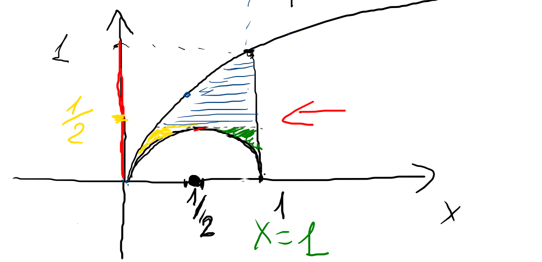
Změňte pořadí integrace u následujících integrálů a přepište je také pomocí transformace do polárních souřadnic (v pořadí $d\varrho\, d\varphi$):

i) $\displaystyle\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{x}} f(x,y)\,dy\,dx,$ ii) $\displaystyle\int_0^1 \int_{\sqrt{2y-y^2}}^{1} f\,dx\,dy.$

$x = y^2 \rightsquigarrow$ pol. $\varrho\cos\varphi = \varrho^2\sin^2\varphi$

$\varrho = \dfrac{\cos\varphi}{\sin^2\varphi}$

$y = \sqrt{x} \rightsquigarrow x = y^2$

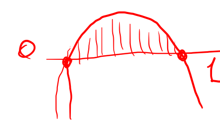


$$\int_0^{1/2}\int_{y^2}^{\frac12-\sqrt{\frac14-y^2}} f\,dx\,dy + \int_0^{1/2}\int_{\frac12+\sqrt{\frac14-y^2}}^{1} f\,dx\,dy + \int_{1/2}^{1}\int_{y^2}^{1} f\,dx\,dy$$

$$\int_0^{\pi/4}\int_{\cos\varphi}^{\frac{1}{\cos\varphi}} f(\varrho\cos\varphi,\varrho\sin\varphi)\,\varrho\,d\varrho\,d\varphi + \int_{\pi/4}^{\pi/2}\int_{\cos\varphi}^{\frac{\cos\varphi}{\sin^2\varphi}} f(\varrho\cos\varphi,\varrho\sin\varphi)\,\varrho\,d\varrho\,d\varphi$$

$y = \sqrt{x - x^2}$

$y = x - x^2$

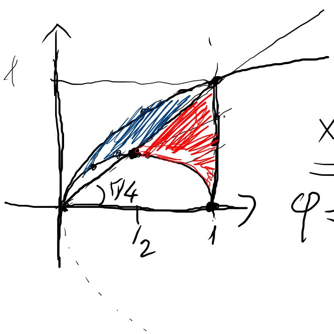


$$\begin{cases} y \geq 0 \\ 0 \leq x \leq 1 \\ x^2 - x + y^2 = 0 \end{cases}$$

$\left(x - \frac12\right)^2 + y^2 = \frac14$

$x - 1/2 = \pm\sqrt{\frac14 - y^2}$

$x = \frac12 - \sqrt{\frac14 - y^2}$ nebo $x = \frac12 + \sqrt{\frac14 - y^2}$



$x = 1 \rightsquigarrow \varrho\cos\varphi = 1,\ \varrho = \dfrac{1}{\cos\varphi}$

$x^2 + y^2 = x \rightsquigarrow$ pol. $\varrho^2 = \varrho\cos\varphi$

$\varrho = \cos\varphi$

$\varphi = 0$